

A new treatment for some periodic Schrödinger operators

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Abstract

We revise some aspects of the asymptotic solution for the eigenvalues for Schrödinger operators with periodic potential, from the perspective of the Floquet theory. In the context of classical Floquet theory, when the periodic potential can be treated as small perturbation we give a new method to compute the asymptotic spectrum. For elliptic potentials a generalized Floquet theory is needed. In order to produce other asymptotic solutions consistent with known results, new relations for the Floquet exponent and the monodromy of wave function are proposed. Many Schrödinger equations of this type, such as the Hill's equation and the ellipsoidal wave equation, etc., can be treated by this method.

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1 Introduction

Consider the following 1-dimensional stationary Schrödinger equation with periodic potential, i.e. a second order periodic ordinary differential equation

$$(\partial_x^2 - u(x))\psi = \lambda\psi, \quad u(x) = u(x + T). \quad (1)$$

It is applied in many areas, from celestial mechanics to accelerator physics and quantum mechanics. There is a large amount of literatures about the linear problem with periodic coefficient [1, 2, 3, 4, 5, 6, 7, 8]. In this paper we focus on the particular aspect about asymptotic solution for the spectrum λ . By “*asymptotic solution*” we mean a solution expanded as an asymptotic series controlled by a small parameter. The parameter space of equation (1) consists of λ and the coupling strength of $u(x)$ collectively denoted by g . A different asymptotic problem is the asymptotic series of eigenfunction $\psi(x)$ for large complex x .

The basic fact about the solution of (1) is the Floquet theory. There are two linearly independent basic solutions to (1), denoted as $\psi_1(x), \psi_2(x)$. As $\psi_1(x+T)$ and $\psi_2(x+T)$ also satisfy the equation, therefore they must be linear combinations of the basic solutions,

$$\begin{pmatrix} \psi_1(x+T) \\ \psi_2(x+T) \end{pmatrix} = M \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}. \quad (2)$$

The 2×2 nonsingular matrix M does not depend on the base point x , it is called the *monodromy matrix*. For Eq. (1) the Wronskian of ψ_1, ψ_2 are constant, so we have $\det M = 1$. Therefore the two eigenvalues of M can be written as $e^{\pm i\nu T}$, they are called the Floquet multipliers. The *Floquet exponent* ν is a function of the eigenvalue and couplings of the potential, $\nu = \nu(\lambda, g)$. In quantum physics ν is called the quasimomentum, and λ is (minus of) the energy, stable solution exists only for real ν . It is a principle problem to find the dispersion relation $\lambda(\nu, g)$ which is the spectral solution of (1). A commonly used method to determine the relation of ν and λ is Hill's method using the infinite determinant. For most periodic potentials $u(x)$, when the parameters take generic value it is impossible to write down an explicit analytical solution. However, it is possible to obtain asymptotic solutions. If the leading order term and the small expansion parameter are known, we can derive the subleading terms from the relation obtained from the Hill's determinant.

This problem has been a classical topic in differential equation and quantum theory. However, at least after reading the books [1, 2, 3, 4, 5, 6, 7, 8], it seems that there are some gaps on this topic. The Floquet theory introduced above can be referred as *classical Floquet theory* as it is a well understood topic for the case of real singly-periodic potential. If the potential is a periodic function of more general type, for example an elliptic function, is there an analogous theory? Generally speaking, not much is known about the Floquet theory for elliptic potential and its relation to the spectral problem. Consider the *ellipsoidal wave equation* for example. From the general consideration that, when the kinetic energy is very large the potential can be treated as small perturbation, i.e. $\lambda \gg \kappa$ where κ is the characteristic strength of the potential (which means the dominant one among all g or certain "average" of all g), an asymptotic spectral solution should exist. Its existence can be inferred also from the relation of the ellipsoidal wave equation and the Lamé equation/Mathieu equation whose large λ spectrum are already known, see e.g. [6, 7, 8] and [9, 10]. However, it seems such large energy (weak coupling) asymptotic solution has not been given for the ellipsoidal wave equation. On the other hand, another asymptotic solution has been obtained sometime ago [11] which gives the spectrum of small perturbation at a stationary point x_* of the potential, i.e. $\lambda = -u(x_*) + \text{perturbation}$, with "perturbation" $\ll \kappa$. But for the small energy (strong coupling) asymptotic solution its connection to the Floquet theory has not been clarified.

In this paper we provide some new results concerning the missing parts mentioned above. In the Section 2 we give a new method for computing the asymptotic solution for large λ , applicable to many periodic spectral problems. In the Section 3 we provide a few examples, including the Hill's equation, the ellipsoidal wave equation and the Heun equation in the elliptic form, to demonstrate the method. We obtain the large energy asymptotic spectra for these equations. In the Section 4 we provide a relation between the small energy spectral solution and the doubly-periodic Floquet theory for elliptic potentials. We show that for a Schrödinger equation with elliptic potential the monodromy of the wave function along each period gives an asymptotic solution.

This paper is motivated by our previous works attempting to examine in detail a few simple examples of the Gauge/Bethe correspondence, proposed by Nekrasov and Shatashvili [12], where the infrared dynamics of some quantum gauge theories is related to the spectral problem of stationary Schrödinger equation with periodic potentials. Some results presented in this paper are still puzzling from the perspective of mathematical theory, albeit they are supported by solid computation and consistent with results already known. We hope the results presented in this paper be useful for further clarification.

2 Large eigenvalue perturbation

In this section our strategy is to use the Floquet theorem to compute $\nu(\lambda)$ for large λ . The eigenvalue relation $\lambda = \lambda(\nu)$ is the reverse of the Floquet exponent $\nu(\lambda)$, therefore if we can compute the monodromy of the wave function along the period then we obtain the eigenvalue expansion.

There is a general relation of the monodromy and the periodic potential. Write the wave function as $\psi(x) = \exp(\int^x v(y)dy)$, then according to the (2) the Floquet exponent is given by

$$i\nu = \frac{1}{T} \int_{x_0}^{x_0+T} v(x)dx. \quad (3)$$

We have picked the positive sign of $e^{\pm i\nu T}$, to obtain the result for the other sector we just change the sign of ν . Substitute the wave function into the Schrödinger equation (1), we get the relation

$$v_x + v^2 = u + \lambda. \quad (4)$$

We use the notation $u_x = \partial_x u$, $u_{xx} = \partial_x^2 u$, \dots , etc. Therefore in order to find all possible asymptotic spectral expansions $\lambda(\nu)$ we can first find all possible asymptotic solutions for $v(x)$ from the equation (4) in the parameter space of λ, g , and then check if the integration (3) gives an asymptotic series. For some potentials it is possible to find many asymptotic

solutions for $v(x)$ from (4), but some solutions may not lead to an asymptotic series after performing the integration of (3).

Now we assume λ is large, therefore $\frac{1}{\sqrt{\lambda}}$ is a natural expansion parameter, then we can expand $v(x)$ by

$$v(x) = \sqrt{\lambda} + \sum_{\ell=1}^{\infty} \frac{v_{\ell}(x)}{(\sqrt{\lambda})^{\ell}}. \quad (5)$$

Substitute the expansion back to (4), we can solve $v_{\ell}(x)$ order by order,

$$\begin{aligned} v_1 &= \frac{1}{2}u, & v_2 &= -\frac{1}{4}u_x, & v_3 &= -\frac{1}{8}(u^2 - u_{xx}), \\ v_4 &= \frac{1}{16}(2u^2 - u_{xx})_x, & v_5 &= \frac{1}{32}(2u^3 + u_x^2 + (u_{xxx} - 6uu_x)_x), \quad \text{etc.} \end{aligned} \quad (6)$$

As $u(x)$ and its derivatives are periodic, we can abandon all terms of total derivative in v_{ℓ} , and especially the even terms do not contribute, $\int_{x_0}^{x_0+T} v_{2\ell} dx = 0$.

We may recognize v_k are Hamiltonian densities of the KdV hierarchy, and relation (4) is the Miura transformation. Indeed, the procedure above is the same as in the KdV theory [13]. This fact has been noticed in e.g. [14], but in their treatment this formalism was not really used. As we would show below this formalism is very useful for computation if the potential is periodic, because the integration in (3) is along the period. We emphasize that for general periodic potentials there is no known direct relation with the KdV theory, and the *formal* connection to the KdV theory is only helpful for computation. However, there are some potentials $u(x)$ with the special choice of coupling strength which solve some higher order generalized stationary KdV hierarchy equations associated to the Hamiltonians given by $\int v_{2\ell-1} dx$. These special potentials include the Lamé potential and the Darboux-Treibich-Verdier potential with triangular number coupling constants, see e.g. [15, 16, 17]

Denote the nonzero integrations by $\varepsilon_{\ell} = \frac{1}{T} \int_{x_0}^{x_0+T} v_{2\ell-1} dx$, it depends on the parameters of the potential but not on x_0 . Then from (3) and (5) we have the relation

$$i\nu = \sqrt{\lambda} + \sum_{\ell=1}^{\infty} \frac{\varepsilon_{\ell}}{(\sqrt{\lambda})^{2\ell-1}}. \quad (7)$$

For many periodic potentials it is very straightforward to explicitly compute ε_{ℓ} because $v_{2\ell-1}$ are polynomials of $u(x)$ and its derivatives. In this way we obtain the asymptotic expansion of $\nu(\lambda)$. Reverse the relation, we get the asymptotic expansion for the eigenvalue,

$$\lambda = -\nu^2 + \sum_{l=0}^{\infty} \frac{\lambda_l}{\nu^{2l}}, \quad (8)$$

with $\lambda_0 = -2\varepsilon_1$, $\lambda_1 = \varepsilon_1^2 + 2\varepsilon_2$, $\lambda_2 = -2(\varepsilon_1^3 + 3\varepsilon_1\varepsilon_2 + \varepsilon_3)$, \dots . The large λ (therefore large ν^2) expansion (8) is actually degenerate for $\pm\nu$.

Compared to the method of Hill's determinant where the relation of ν and λ holds for generic value, in our treatment we have assumed the large λ asymptotic form for $v(x)$ from the beginning of computation, in formula (5). Therefore, the procedure from (5) to (8) only applies to this asymptotic expansion region, however, it is very efficient from the computational point view. In fact, the method to derive eigenfunction from the solution of the relation (4) not only applies to the case of large energy solution, it also applies to the case of small energy solution discussed in the Section 4.

After obtaining the asymptotic eigenvalue, we can continue to find the corresponding asymptotic eigenfunction. The following are also standard results of the Floquet theory, see e.g.[5]. If $e^{i\nu T} \neq \pm 1$, then there are two independent stable solutions given by $\psi_1(x) = e^{i\nu x}\phi(x)$ and $\psi_2(x) = e^{-i\nu x}\phi(-x)$, with $\phi(x)$ a periodic function, $\phi(x+T) = \phi(x)$. If $e^{i\nu T} = \pm 1$, then there is only one stable solution given by $\psi(x) = e^{i\nu x}\phi(x)$, another independent solution is unstable. As we obtain the solution (5) for $v(x)$, the unnormalized asymptotic eigenfunction can be written in the exponential form $\psi(x) = \exp(\int^x v(y)dy)$ which explicitly depends on λ , then we substitute the asymptotic eigenvalue (8) to obtain the corresponding eigenfunction with explicit dependence on ν .

In the following section we present a few examples to show how to use this method to obtain the asymptotic eigenvalue for some periodic Schrödinger equations.

3 Some examples

3.1 Hill's equation

We start with an easy example to demonstrate the method. The Hill's equation often refers to equation of the form (1) with a general real periodic potential. By the Fourier expansion the potential can be represented by a trigonometric polynomial,

$$u(x) = \sum_{n=1}^{\infty} 2\theta_n \cos 2nx, \quad (9)$$

the period is π . The coupling constants are θ_n , in some cases they may be truncated to a finite subset if the approximation is valid. The Hill's equation was used in celestial mechanics to achieve a high-accuracy description of the motion of moon under the influence of earth and sun.

Let us specify to the simple case with $\theta_n = 0$ for $n \geq 3$,

$$u(x) = 2\theta_1 \cos 2x + 2\theta_2 \cos 4x. \quad (10)$$

The resulting equation is called the Whittaker-Hill equation. It arises when we rewrite the 3-dimensional wave equation $\nabla^2 W + \tilde{\chi}^2 W = 0$ in the paraboloidal coordinates and apply

the separation of variables method, the wave equation reduces to three identical equations of Whittaker-Hill type, see [3]. The integration results for ε_ℓ are

$$\varepsilon_1 = 0, \quad \varepsilon_2 = -\frac{1}{4}(\theta_1^2 + \theta_2^2), \quad \varepsilon_3 = \frac{1}{8}(2\theta_1^2 + 8\theta_2^2 + 3\theta_1^2\theta_2), \quad \text{etc}, \quad (11)$$

and then by (8) we obtain

$$\begin{aligned} \lambda = & -\nu^2 - \frac{\theta_1^2 + \theta_2^2}{2\nu^2} - \frac{2\theta_1^2 + 8\theta_2^2 + 3\theta_1^2\theta_2}{4\nu^4} \\ & - \frac{16\theta_1^2 + 256\theta_2^2 + 120\theta_1^2\theta_2 + 5\theta_1^4 + 40\theta_1^2\theta_2^2 + 5\theta_2^4}{32\nu^6} + \dots \end{aligned} \quad (12)$$

3.2 Ellipsoidal wave equation

The more interesting examples are the equation (1) with elliptic potentials. The elliptic potentials have two independent periods, from which we can make the third period. It is questionable whether the classical Floquet theory can be directly applied to all periods, up to now very limited results on this problem has been obtained [3, 4, 5]. As we would show in the rest of the paper, the classical Floquet theory is still valid for one period, but a generalization is needed for other periods.

If we rewrite the 3-dimensional wave equation $\nabla^2 W + \tilde{\chi}^2 W = 0$ in the ellipsoidal coordinates, apply the separation of variables method, then the three identical equations are *ellipsoidal wave equation*, see [3]. Written in the Jacobian form it is

$$\partial_z^2 \psi(z) - (\Delta k^2 \text{sn}^2 z + \Omega k^4 \text{sn}^4 z) \psi(z) = \Lambda \psi(z), \quad (13)$$

where $\Omega \propto \tilde{\chi}^2$, and $\text{sn}(z|k^2)$ is the Jacobian elliptic function with the elliptic modulus k , its quarter periods are the complete elliptic integrals $K(k^2)$ and $iK'(k^2) = iK(1 - k^2)$.

The Weierstrass form is also useful for our purpose,

$$\partial_x^2 \psi(x) - (\alpha_1 \wp(x) + \alpha_2 \wp(x)^2) \psi(x) = \lambda \psi(x), \quad (14)$$

where $\wp(x) = \wp(x; 2\omega_1, 2\omega_2)$ is the Weierstrass elliptic function, ω_1, ω_2 are the two independent half periods, and $\omega_3 = \omega_1 + \omega_2$ also plays a role in this study. The following relations between $x, \wp(x)$ and $z, \text{sn} z$ are used,

$$x = \frac{z + iK'}{(e_1 - e_2)^{1/2}}, \quad \wp(x) = e_2 + (e_3 - e_2) \text{sn}^2 z, \quad (15)$$

where $e_i = \wp(\omega_i)$ and they satisfy $e_1 + e_2 + e_3 = 0$. The relation between periods is $K = (e_1 - e_2)^{1/2} \omega_1$, $iK' = (e_1 - e_2)^{1/2} \omega_2$. The nome of the \wp -function and e_i is $q = \exp(2\pi i \frac{\omega_2}{\omega_1}) = \exp(-2\pi i \frac{K'}{K})$, related to the elliptic modulus k by

$$k^2 = \frac{e_3 - e_2}{e_1 - e_2} = \frac{\vartheta_2^4(q)}{\vartheta_3^4(q)}. \quad (16)$$

The parameters $(\alpha_1, \alpha_2, \lambda)$ are related to $(\Delta, \Omega, \Lambda)$ by

$$\alpha_1 = \Delta - \frac{2e_2\Omega}{e_1 - e_2}, \quad \alpha_2 = \frac{\Omega}{e_1 - e_2}, \quad \lambda = (e_1 - e_2)\Lambda - e_2\Delta + \frac{e_2^2\Omega}{e_1 - e_2}. \quad (17)$$

There is a technical, nevertheless useful fact about the equations with elliptic potential, that the Jacobian form and the Weierstrass form each has an advantage against the other in deriving different asymptotic solutions. The Weierstrass form is more suitable for driving the large λ perturbation given in this section, and the Jacobian form is more suitable for other two perturbations given in the next section.

The large energy asymptotic solution

In the definition of the elliptic function, the two periods $2\omega_1$ and $2\omega_2$ are on equal footing. Does the Floquet theorem apply to each period equally? Do we get two asymptotic spectral solutions from periods $2\omega_1$ and $2\omega_2$, respectively, using the same method in the Section 2? Based on the computational results, given in the Section 4, the answer is not. The classical Floquet theory is valid only for the period $2\omega_1$ and the computation method for large λ solution indeed works. But for the periods $2\omega_2$ and $2\omega_3$ we need a generalized Floquet theory, the corresponding asymptotic spectral solution has a different nature, because the large λ assumption fails. In this section we only deal with the large λ solution, and in the Section 4 we deal with cases associated to periods $2\omega_2$ and $2\omega_3$.

The integrands $v_{2\ell-1}$ contain higher powers of $\wp(x)$ and $\wp'(x)$, where the prime denotes ∂_x , they can be simplified using relations derived from the basic relation $\wp'(x)^2 = 4\wp^3 - g_2\wp - g_3$. The simplified integrands, after discarding total derivative terms, take the form $p_0(g_2, g_3) + p_1(g_2, g_3)\wp(x)$ which is ready for integration, where p_0, p_1 are polynomial functions with arguments g_2, g_3 . The integration results for ε_ℓ are

$$\begin{aligned} \varepsilon_1 &= -\frac{1}{2}\alpha_1\zeta_1 + \frac{1}{24}\alpha_2g_2, \\ \varepsilon_2 &= -\frac{1}{96}\alpha_1^2g_2 + \frac{1}{80}\alpha_1\alpha_2(3g_2\zeta_1 - 2g_3) + \frac{1}{2688}\alpha_2^2(48g_3\zeta_1 - 5g_2^2), \quad \text{etc}, \end{aligned} \quad (18)$$

where ζ_1 is defined by the Weierstrass zeta function $\zeta_1 = \frac{\zeta(\omega_1)}{\omega_1}$, the modular invariants g_2, g_3 are given by $g_2 = -4(e_1e_2 + e_1e_3 + e_2e_3)$, $g_3 = 4e_1e_2e_3$. They also can be rewritten in terms of the Eisenstein series E_2, E_4, E_6 , or in terms of the theta constants $\vartheta_r(q)$, $r = 1, 2, 3, 4$. We denote the Floquet exponent of wave function in Eq. (14) as ν , i.e. $\psi(x + 2\omega_1) = \exp(i2\nu\omega_1)\psi(x)$, then the asymptotical expansion for λ is

$$\begin{aligned} \lambda &= -\nu^2 + \frac{1}{12}(12\alpha_1\zeta_1 - \alpha_2g_2) + \frac{1}{5040\nu^2}[105\alpha_1^2(12\zeta_1^2 - g_2) + 84\alpha_1\alpha_2(2g_2\zeta_1 - 3g_3) \\ &\quad + 10\alpha_2^2(18g_3\zeta_1 - g_2^2)] + \mathcal{O}(\frac{1}{\nu^4}). \end{aligned} \quad (19)$$

The eigenvalue Λ for the equation in Jacobian form can be transformed from λ . However, the definition for the Floquet exponent differs. We use μ to denote the Floquet exponent of wave function in Jacobian form (13), i.e. $\psi(z + 2K) = \exp(i2\mu K)\psi(z)$. Shifting x by $2\omega_1$ is the same as shifting z by $2K$, therefore the phases should be the same, $\nu\omega_1 = \mu K$ therefore we have $\nu = (e_1 - e_2)^{1/2}\mu$. Taking into account the relation in (17), the relation of q and k , we obtain

$$\begin{aligned} \Lambda = & -\mu^2 - \left[\frac{\Delta}{2}k^2 + \frac{\Delta + 6\Omega}{16}k^4 + \frac{\Delta + 2\Omega}{32}k^6 + \frac{41\Delta + 70\Omega}{2048}k^8 + \dots \right] \\ & - \frac{1}{\mu^2} \left(\frac{\Delta^2}{32}k^4 + \frac{\Delta\Omega}{16}k^6 - \frac{\Delta^2 - 8\Delta\Omega - 136\Omega^2}{4096}k^8 - \frac{(\Delta - 4\Omega)(\Delta + 2\Omega)}{4096}k^{10} + \dots \right) + \dots \end{aligned} \quad (20)$$

Apparently the expansion (19) is more compact, therefore Eq. (14) is a better form for the large eigenvalue asymptotic solution.

Taking the limit $\Omega \rightarrow 0$ we recover the results for the Lamé equation, already treated in [18], see also [19] and references in. The Lamé equation comes from the same procedure of solving the Laplace equation $\nabla^2 W = 0$ in the ellipsoidal coordinates. Taking the limit $k^2 \rightarrow 0$ while keeping $\Delta k^2 \rightarrow -4\theta_1 - 16\theta_2$, $\Omega k^4 \rightarrow 16\theta_2$ we recover the result for the Whittaker-Hill equation. Taking a further limit $\theta_2 \rightarrow 0$ we recover the result for the Mathieu equation.

3.3 Heun equation in elliptic form

A generalization of the Lamé equation is the Heun equation in the elliptic form. The term Heun equation often refers to its normal form where the Fushian property of the equation becomes apparent [20, 8]. In the Jacobian form given by Darboux [21] the equation is

$$\partial_z^2 \psi(z) - \left(b_0 k^2 \text{sn}^2 z + b_1 k^2 \frac{\text{cn}^2 z}{\text{dn}^2 z} + b_2 \frac{1}{\text{sn}^2 z} + b_3 \frac{\text{dn}^2 z}{\text{cn}^2 z} \right) \psi(z) = \Lambda \psi(z). \quad (21)$$

In the Weierstrass form it is

$$\partial_x^2 \psi(x) - \sum_{s=0}^3 b_s \wp(x + \omega_s) \psi(x) = \lambda \psi(x), \quad (22)$$

where $\omega_0 = 0$. The multi-component potential in (22) is the so-called Treibich-Verdier potential, known for its role in the theory of “elliptic soliton” for KdV hierarchy [17]. To relate (21) and (22), the coordinates transformation is the same as in (15), Λ and λ are related by

$$\Lambda = \frac{\lambda + e_2 \left(\sum_{s=0}^3 b_s \right)}{e_1 - e_2}. \quad (23)$$

As in the previous example, the equation in the form (22) is more suitable for the large λ expansion. In the process of computing ε_ℓ we need to simplify the integrands by repeatedly using relations about the \wp -function with the same argument, already used in the previous subsection, and some other relations such as,

$$\begin{aligned}
\wp(x)\wp(x+\omega_i) &= e_i[\wp(x) + \wp(x+\omega_i)] + e_i^2 + e_j e_k, \\
\wp(x+\omega_i)\wp(x+\omega_j) &= e_k[\wp(x+\omega_i) + \wp(x+\omega_j)] + e_k^2 + e_i e_j, \\
\wp'(x)\wp'(x+\omega_i) &= -4(e_i - e_j)(e_i - e_k)[\wp(x) + \wp(x+\omega_i) + e_i], \\
\wp'(x+\omega_i)\wp'(x+\omega_j) &= -4(e_i - e_k)(e_j - e_k)[\wp(x+\omega_i) + \wp(x+\omega_j) + e_k], \\
&\text{etc,}
\end{aligned} \tag{24}$$

in the above index $i, j, k \in \{1, 2, 3\}$ and $i \neq j \neq k$. The integrands are finally simplified to the form $p_0(e_{1,2,3}) + \sum_s p_1^{(s)}(e_{1,2,3})\wp(x+\omega_s)$, $s = 0, 1, 2, 3$, ready for integration. Then we obtain

$$\begin{aligned}
\varepsilon_1 &= -\frac{1}{2}\left(\sum_{s=0}^3 b_s\right)\zeta_1, \\
\varepsilon_2 &= \frac{1}{24}\left(\sum_{s=0}^3 b_s^2\right)(e_1 e_2 + e_1 e_3 + e_2 e_3) - \frac{1}{4}[(b_0 b_1 + b_2 b_3)(e_1^2 + e_2 e_3 - 2e_1 \zeta_1) \\
&\quad + (b_0 b_2 + b_1 b_3)(e_2^2 + e_1 e_3 - 2e_2 \zeta_1) + (b_0 b_3 + b_1 b_2)(e_3^2 + e_1 e_2 - 2e_3 \zeta_1)], \quad \text{etc.}
\end{aligned} \tag{25}$$

In the above expressions ζ_1 is related to E_2 , and e_1, e_2, e_3 are related to E_4, E_6 . They generalize the results for the Lamé equation presented in [18].

It is straightforward to compute λ (and Λ) from ε_ℓ , here we do not give its explicit expression. Using these results we can verify the relation between the spectrum of Schrödinger operator and the effective action of the deformed N=2 supersymmetric gauge theory model, in the spirit of Gauge/Bethe correspondence [12], up to arbitrary higher order expansion. This computation completes the attempts in [22] where the leading order expansion was examined ¹.

¹We remind readers the issue of notation. For the Jacobian elliptic functions in this paper we use k to denote the elliptic modulus, while in [22] we used q . For the Weierstrass elliptic functions in this paper we use q to denote the nome, while in [22] we used p . The reason for this change is that in [22] we tried to respect the convention of gauge theory literature while in this paper we adopt convention of mathematical literature. In the computation the following relation is used, in the notation of this paper, $k^2 \frac{\partial}{\partial k^2} \mathcal{F}(k^2) = \frac{2}{e_1 - e_2} q \frac{\partial}{\partial q} \mathcal{F}(q)$, where \mathcal{F} is the deformed gauge theory prepotential and k^2 is the instanton counting parameter for SU(2) $N_f = 4$ super-QCD model discussed in [22].

4 On doubly-periodic Floquet theory

4.1 Spectral problem for elliptic potentials

We have shown that for elliptic potentials the large λ asymptotic solution is always related to the monodromy along the periodic $2\omega_1$. So what is the role for $2\omega_2$ and $2\omega_3$? This question is related to the generalized Floquet theory for elliptic function, the so-called *doubly-periodic Floquet theory*, which only has been occasionally discussed during the past, e.g. in [23, 24, 25, 26, 27]. Some new features due to the complex nature of the elliptic function arise, make the extension nontrivial. Among the limited results that already exist on this topic, it seems that there is not an explicit statement about the relation of monodromy along $2\omega_2$, $2\omega_3$ and the spectrum of the equation. In this section we give a few examples to show that the monodromy of $v(x)$ along $2\omega_2$ and $2\omega_3$ indeed play a role in the spectral problem, they are related to two other asymptotic solutions that differ from (20) given above.

Therefore the problem we are trying to answer is related to the complete characterization of all asymptotic spectra for elliptic potentials. For such a Schrödinger operator the spectral solution λ is controlled by the characteristic coupling strength of potential κ , or more precisely by the ratio $\frac{\nu}{\kappa}$, often it has no analytical expression. How does the relation $\lambda(\nu, \kappa)$ vary when we turn the value of $\frac{\nu}{\kappa}$? When $\lambda(\nu, \kappa)$ can be represented by an asymptotic series? The answer is not obvious at all. In the literature it is even not systematically studied how many asymptotic solutions there are for an elliptic potential.

It is necessary to explain the meaning of “spectrum” for a complex potential. The elliptic potentials are meromorphic function defined on the complex plane, therefore, they are not the most suitable examples for quantum mechanics. Instead their appearance in quantum field theory looks more natural, where the complex valued spectrum of Schrödinger operator is explained in a very different way. Indeed, in the context of Gauge/Bethe correspondence [12] the spectral solution of elliptic potentials nicely fits into the theory of 4-dimensional quantum gauge theory. For the Lamé potential, due to its connection with a typical Seiberg-Witten gauge theory model [28, 29], the idea of using elliptic curve is very helpful for the analysis, there is a one-to-one correspondence between the asymptotic solutions and the monodromy of wave function along $2\omega_i, i = 1, 2, 3$ [19]. Upon a careful examination, the complete spectral solutions are precisely related to nonperturbative and duality properties of the low energy effective gauge theory. Another related context for the elliptic potential is the algebraic integrable theory, see e.g. [13], albeit neither the questions mentioned above have been seriously addressed there.

4.2 Lamé equation

The Lamé potential is the first example that motivates us to revise the doubly-periodic Floquet theory from a new perspective. It is $u(x) = \Delta \wp(x)$ in the Weierstrass form, or $u(z) = \Delta k^2 \text{sn}^2 z$ in the Jacobian form. We briefly review the result to give a general picture about the answer, the details are already given in [19].

The first fact is about the stationary points of the potential. There are three stationary points for the potential given by the solutions of $\partial_x \wp(x) = 0$, they are at $x_* = \omega_1, \omega_2, \omega_3$ where we have $u(x_*) = e_1 \Delta, e_2 \Delta, e_3 \Delta$. In the Jacobian form the three stationary points are given by the solutions of $\partial_z \text{sn}^2 z = 0$, they are at $z_* = K + iK', 0, K$, and $u(z_*) = \Delta, 0, \Delta k^2$. The information about these stationary points does not tell us what are the possible asymptotic solutions, the following facts entirely come from computation [19] (see also [22]).

It turns out that each stationary point is associated to an asymptotic expansion for λ . The stationary point at $x_* = \omega_1$ (i.e. at $z_* = K + iK'$) is associated to the large λ solution. The equation in the Weierstrass form is better for computation. The leading order energy comes from the quasimomentum, $\lambda = -\nu^2 + \dots$, the potential can be treated as small perturbation, therefore we have $\nu \sim \sqrt{-\lambda} \gg \kappa \sim \Delta$. The relation $\nu(\lambda)$ is given by the monodromy along the period $2\omega_1$ as in the formula (3). This is well described by the classical Floquet theory, the asymptotic solution can be treated by the method given in the Section 2, see [18].

The other two stationary points are related to two other asymptotic solutions, to study them we need the generalized Floquet theory. The equation in the Jacobian form is better for computation. In these cases the quasimomentum is small compared to the scale of potential which means $\mu \ll \kappa \sim \Delta k^2$. The solution $\Lambda \sim 0 + \dots$ (i.e. $\lambda \sim -e_2 \Delta + \dots$) is a perturbation at $z_* = 0$ (i.e. at $x_* = \omega_2$), here we have $\Lambda \ll \Delta k^2$. The relation $\mu(\Lambda)$ is given by the monodromy of wave function along the period $2iK'$ (i.e. $2\omega_2$). A key point is that the naive definition of the Floquet exponent $\psi(z + 2iK') = \exp(-2\mu K')\psi(z)$ is not right. If we want to produce the correct asymptotic expansion that is already derived by other method in [11], then a modification is needed, the right relation is $\psi(z + 2iK') = \exp(\mu\pi)\psi(z)$. The solution $\Lambda \sim -\Delta k^2 + \dots$ (i.e. $\lambda \sim -e_3 \Delta + \dots$) is a perturbation at $z_* = K$ (i.e. at $x_* = \omega_3$). The subleading terms are denoted by $\tilde{\Lambda}$, i.e. $\Lambda = -\Delta k^2 + \tilde{\Lambda}$ with $\tilde{\Lambda} \ll \Delta k^2$. Then the relation $\mu(\tilde{\Lambda})$ is given by the monodromy along the period $2(K + iK')$ (i.e. $2\omega_3$). Again the classical Floquet theory fails and the correct definition of Floquet exponent is given by $\psi(z + 2K + 2iK') = \exp(\frac{\mu\pi}{ik'})\psi(z)$, with the complementary module k' satisfying the relation $k'^2 + k^2 = 1$.

While we do not have a mathematical theory to explain why the monodromies along three periods are in one-to-one correspond with three asymptotic solutions, nevertheless a

physical explanation was given [19]. Viewed from the Gauge/Bethe correspondence [12], the spectral problem of the Lamé operator is roughly the same problem about the low energy effective theory of gauge theory model. The monodromies along different periods are related by electro-magnetic duality of the effective gauge theory, in the spirit of Seiberg-Witten theory [28, 29]. For the gauge theory model there is an asymptotic solution in each duality frame, hence for the Lamé operator there is an asymptotic solution related to the monodromy along each period.

4.3 Ellipsoidal wave equation

Now we turn to the next example. It seems that this method found for the Lamé equation also applies to other equations with elliptic coefficient. The ellipsoidal wave equation is non-Fuchsian, has no known relation to gauge theory, nevertheless, it can be treated in a similar way.

The stationary points of the Lamé potential are also the stationary points of the potential $u(z) = \Delta k^2 \text{sn}^2 z + \Omega k^4 \text{sn}^4 z$. The monodromy along $2K$ (i.e. $2\omega_1$) gives the large Λ asymptotic solution, this is the result given in (20). In the following we give the computation details to demonstrate that the monodromies along $2iK'$ and $2K + 2iK'$ give other asymptotic eigenvalues, one of them was already obtained by another method [11].

The first small energy asymptotic solution

The equation in the Jacobian form (13) is more suitable for this asymptotic solution. We assume $\Delta k^2 \text{sn}^2 z$ is the dominant term of the potential, i.e. $\kappa = \Delta k^2$, the other term $\Omega k^4 \text{sn}^4 z$ is a small perturbation. At the point $z_* = 0$, we have $u(z_*) = 0$, therefore Λ is the perturbative energy. The parameters satisfy $\Omega k^4, \Lambda \ll \Delta k^2$. We shall find the asymptotic expansion for the integrand $v(z)$ from the relation $v_z + v^2 = u + \Lambda$. Now the expansion parameter should be $\Delta^{\frac{1}{2}} k$, or equivalently $\Delta^{\frac{1}{2}}$, with $v(z)$ expanded as

$$v(z) = \sqrt{\Delta} v_{-1}(z) + v_0(z) + \sum_{\ell=1}^{\infty} \frac{v_{\ell}(z)}{(\sqrt{\Delta})^{\ell}}, \quad (26)$$

then $v_{\ell}(z)$ can be recursively solved. The even terms $v_{2\ell}, \ell = 0, 1, 2, \dots$ are total derivatives, they do not contribute in the final periodic integration (28). The nonzero contributions come

from $v_{2\ell-1}$, the first few are

$$\begin{aligned}
v_{-1} &= k \operatorname{sn} z, \\
v_1 &= \frac{1}{2} \Omega k^3 \operatorname{sn}^3 z + \frac{1}{8} k \operatorname{sn} z + \frac{1+k^2+4\Lambda}{8k \operatorname{sn} z} - \frac{3}{8k \operatorname{sn}^3 z}, \\
v_3 &= -\frac{1}{8} \Omega^2 k^5 \operatorname{sn}^5 z + \frac{7}{16} \Omega k^3 \operatorname{sn}^3 z - \frac{1}{128} k(1+40\Omega(1+k^2)+32\Lambda\Omega) \operatorname{sn} z \\
&\quad + \frac{7(1+k^2)+28\Lambda+12\Omega k^2}{64k \operatorname{sn} z} + \dots.
\end{aligned} \tag{27}$$

Then we come to the issue of relating the monodromy of wave function along period $2iK'$ to the Floquet exponent μ . According to the classical Floquet theory, the relation should be $\int v(z)dz = i\mu \times 2iK' = -2\mu K'$. Indeed we can use this relation as the definition of the Floquet exponent. However, the corresponding asymptotic spectral solution has already been obtained by a different method in [11], the result suggests that the classical Floquet theory cannot be directly applied to the period $2iK'$. We find the correct relation between the period integral and the Floquet exponent is

$$\mu = \frac{1}{\pi} \int_{z_0}^{z_0+2iK'} v(z) dz. \tag{28}$$

This modified relation is the same as that for the Lamé equation, it leads to the asymptotic solution given in (30) which is the one obtained in [11]. We believe this is how the classical Floquet theory should be generalized for elliptic potentials, albeit at the moment we do not have a rigorous proof.

The integration formulae for $v_{2\ell-1}$ can be found in [30]. If we denote $I_m = \int_{z_0}^{z_0+2iK'} \operatorname{sn}^m z dz$, then we have $I_m = 0$ for $m = 1, 3, 5, \dots$, and the remaining non-vanishing I_{-m} are

$$I_{-1} = i\pi, \quad I_{-3} = i\pi \frac{1+k^2}{2}, \quad I_{-5} = i\pi \frac{3+2k^2+3k^4}{8}, \quad \text{etc.} \tag{29}$$

They have been used in the previous related computation for the Lamé equation in [19], we give more details in the Appendix A. Reverse the series $\mu = \mu(\Lambda)$ we reproduce the asymptotic expansion given in [11],

$$\begin{aligned}
\Lambda &= -i2\Delta^{\frac{1}{2}}k\mu - \frac{1}{2^3}(1+k^2)(4\mu^2-1) \\
&\quad - \frac{i}{2^5\Delta^{\frac{1}{2}}k}[(1+k^2)^2(4\mu^3-3\mu)-4k^2(4\mu^3-5\mu)] \\
&\quad + \frac{1}{2^{10}\Delta k^2}[(1+k^2)(1-k^2)^2(80\mu^4-136\mu^2+9)+384\Omega k^4(4\mu^2-1)] \\
&\quad + \frac{i}{2^{13}\Delta^{\frac{3}{2}}k^3}[(1+k^2)^4(528\mu^5-1640\mu^3+405\mu)-24k^2(1+k^2)^2(112\mu^5-360\mu^3+95\mu) \\
&\quad + 16k^4(144\mu^5-520\mu^3+173\mu)-512\Omega k^4(1+k^2)(4\mu^3-11\mu)] + \dots.
\end{aligned} \tag{30}$$

In this expression we use notations slightly different from that in [11], in order to keep consistent with our previous paper [19] where the difference is explained.

The second small energy asymptotic solution

The second small energy expansion is a perturbation at $z_* = K$ where $\text{sn}^2 z_* = 1$, therefore $u(z_*) = \Delta k^2 + \Omega k^4$. Similar to the treatment in [19] we set $\Lambda = -\Delta k^2 - \Omega k^4 + \tilde{\Lambda}$, where $\tilde{\Lambda}$ is the perturbative energy around the local minimum of potential. The equation becomes

$$\partial_z^2 \psi(z) + [\Delta k^2 \text{cn}^2 z + \Omega k^4 \text{cn}^2 z(2 - \text{cn}^2 z)]\psi(z) = \tilde{\Lambda} \psi(z). \quad (31)$$

The parameters satisfy $\Omega k^4, \tilde{\Lambda} \ll \Delta k^2$, therefore similar to the case of the previous solution, we choose $\Delta^{\frac{1}{2}}$ as the expansion parameter and expand the integrand $v(z)$ as

$$v(z) = i \sum_{\ell=-1}^{\infty} \frac{v_{\ell}(z)}{(\sqrt{\Delta})^{\ell}}. \quad (32)$$

It includes the factor i because the potential in this case is $u(z) = -\Delta k^2 \text{cn}^2 z - \Omega k^4 \text{cn}^2 z(2 - \text{cn}^2 z)$. From the relation $v_z + v^2 = u + \tilde{\Lambda}$ we obtain the expressions for $v_{\ell}(z)$. The even terms $v_{2\ell}, \ell = 0, 1, 2, \dots$ are again total derivatives and do not contribute to the final integration of (34). Other $v_{2\ell-1}$ contribute non-vanishing integrations, the first few are

$$\begin{aligned} v_{-1} &= k \text{cn} z, \\ v_1 &= -\frac{1}{2} \Omega k^3 \text{cn}^3 z + \frac{k(1 + 8\Omega k^2)}{8} \text{cn} z - \frac{1 - 2k^2 + 4\tilde{\Lambda}}{8k \text{cn} z} + \frac{3(1 - k^2)}{8k \text{cn}^3 z}, \\ v_3 &= -\frac{1}{8} \Omega^2 k^5 \text{cn}^5 z - \frac{\Omega k^3(7 - 8\Omega k^2)}{16} \text{cn}^3 z - \frac{1}{128} k(1 + 40\Omega - 64\Omega k^2 + 32\tilde{\Lambda}\Omega + 64\Omega^2 k^4) \text{cn} z \\ &\quad - \frac{7(1 - 2k^2) + 28\tilde{\Lambda} - 4\Omega k^2(5 - 7k^2) - 32\tilde{\Lambda}\Omega k^2}{64k \text{cn} z} + \dots \end{aligned} \quad (33)$$

Concerned the issue of relating the monodromy of the wave function along period $2K + 2iK'$ and the Floquet exponent μ , similar to the case of the first small energy asymptotic expansion, the classical Floquet theory is invalid. Although the corresponding asymptotic expansion presented below in (36) has not been given in other literature, there is the requirement of consistent with other known results. We find the correct relation is given by

$$\mu = \frac{ik'}{\pi} \int_{z_0}^{z_0 + 2K + 2iK'} v(z) dz. \quad (34)$$

This relation gives the asymptotic expansion (36) consistent with all known results, especially in the limits of $\Omega \rightarrow 0$ (the Lamé potential), and in the limit $\Omega \rightarrow 0, k \rightarrow 0$, with Δk^2 fixed (the Mathieu potential).

The integration formulae for $v_{2\ell-1}$ we need in this case are denoted by $J_m = \int_{z_0}^{z_0+2K+2iK'} \text{cn}^m z dz$, we have $J_m = 0$ for $m = 1, 3, 5, \dots$, and

$$J_{-1} = -i\pi \frac{1}{k'}, \quad J_{-3} = -i\pi \frac{1-2k^2}{2k'^3}, \quad J_{-5} = -i\pi \frac{3-8k^2+8k^4}{8k'^5}, \quad \text{etc}, \quad (35)$$

they have been used in [19] too. After getting the asymptotic series $\mu = \mu(\tilde{\Lambda})$ we reverse it, the final asymptotic expansion for the spectral relation $\Lambda = -\Delta k^2 - \Omega k^4 + \tilde{\Lambda}(\mu)$ is

$$\begin{aligned} \Lambda = & -\Delta k^2 - \Omega k^4 + i2\Delta^{\frac{1}{2}}k\mu + \frac{1}{2^3}(1-2k^2)\left(\frac{4\mu^2}{k'^2} + 1\right) \\ & + \frac{i}{\Delta^{\frac{1}{2}}k} \left\{ \frac{1}{2^5} \left[\frac{(1-2k^2)^2}{k'} \left(\frac{4\mu^3}{k'^3} + \frac{3\mu}{k'} \right) + 4k^2k' \left(\frac{4\mu^3}{k'^3} + \frac{5\mu}{k'} \right) \right] + 2\Omega k^4\mu \right\} \\ & - \frac{1}{\Delta k^2} \left[\frac{1-2k^2}{2^{10}k'^2} \left(\frac{80\mu^4}{k'^4} + \frac{136\mu^2}{k'^2} + 9 \right) - \frac{3}{8}\Omega k^4(4\mu^2 + k'^2) \right] \\ & - \frac{i}{\Delta^{\frac{3}{2}}k^3} \left\{ \frac{1}{2^{13}} \left[\frac{(1-2k^2)^4}{k'^3} \left(\frac{528\mu^5}{k'^5} + \frac{1640\mu^3}{k'^3} + \frac{405\mu}{k'} \right) + \frac{24k^2(1-2k^2)^2}{k'} \left(\frac{112\mu^5}{k'^5} + \frac{360\mu^3}{k'^3} + \frac{95\mu}{k'} \right) \right] \right. \\ & + 16k^4k' \left(\frac{144\mu^5}{k'^5} + \frac{520\mu^3}{k'^3} + \frac{173\mu}{k'} \right) \left. \right] + \frac{\Omega k^4}{2^5k'^4} [4(4k^4 - 6k^2 + 3)\mu^3 + k'^2(36k^4 - 58k^2 + 25)\mu] \\ & + \Omega^2 k^8 \mu \} + \dots \end{aligned} \quad (36)$$

We write the expansion in a form easy to see its connection to the eigenvalue of Lamé equation, in the limit $\Omega \rightarrow 0$. The Ω -independent part in expansion (36) has an interesting relation to the expansion (30), which is already explained in [19].

The potential of ellipsoidal wave equation actually has more stationary points z_* given by the solutions of $2\Omega k^2 \text{sn}^2 z_* + \Delta = 0$. Are they associated to new asymptotic spectral solutions? We do not have a definite answer to this question. However, even new asymptotic solutions exist they are unlikely given by the monodromy along a period, therefore not in the scope of Floquet theory.

4.4 Darboux-Treibich-Verdier potential

The Heun equation is also a Fushian equation, in the Section 3 using its elliptic form we have shown the classical Floquet theory indeed applies to the period $2\omega_1$ (i.e. $2K$) and gives the large λ spectral solution. We believe the doubly-periodic Floquet theory is applicable to other periods $2iK'$ and $2K+2iK'$ (i.e. $2\omega_2$ and $2\omega_3$), similar to the case in the Subsection 4.3. However, there is a technical difficulty for the computation of the higher order perturbation, explained below.

From the result of [22] there are six stationary points z_* for the Darboux-Treibich-Verdier potential given by the solutions of $\partial_z u(z) = 0$, each corresponds to an asymptotic spectral solution. Four of them are related to the large λ asymptotic solution given by the result

presented in the Subsection 3.3. The other two stationary points are at $\Lambda_* = -u(z_*) \sim \pm[(b_0 - b_1)(b_2 - b_3)]^{1/2}k + \mathcal{O}(k^2)$, where Λ_* is the same order of the geometric average of the potential terms $(b_0 b_1 b_2 b_3)^{1/4}k$, they are related to the remaining two asymptotic solutions. The corresponding Floquet exponents are given by their relation to the monodromy as in formulae (28) and (34). Then we can rewrite the eigenvalue as $\Lambda = \delta + \Lambda_*$ where δ is the small perturbation around the local minimum, $\delta \ll \Lambda_*$. Now the problem is to find a proper expansion for the integrand $v(z)$ suitable for integration from the relation

$$v_z + v^2 = \delta + \Lambda_* + b_0 k^2 \text{sn}^2 z + b_1 k^2 \frac{\text{cn}^2 z}{\text{dn}^2 z} + b_2 \frac{1}{\text{sn}^2 z} + b_3 \frac{\text{dn}^2 z}{\text{cn}^2 z}. \quad (37)$$

In this case the potential is not dominated by a single term, instead every term equally contributes to the potential, moreover, Λ_* is a constant of the same order of the “averaged” potential. These facts make it technically difficult to find the proper expansion for $v(z)$ whose expansion coefficients can be recursively solved.

Nevertheless, there is a simple way to see evidences that the doubly-periodic Floquet theory indeed applies to the Darboux-Treibich-Verdier potential. In [22] using the normal form of the Heun equation, the monodromy is computed as a WKB series up to the leading order, the integration contours are determined by the roots of a quadratic polynomial. It is an algebraic exercise, despite a bit tedious, to show that the contours in the normal form are exactly in correspondence with the periodic integrals along $2\omega_i, i = 1, 2, 3$ (i.e. $2K, 2iK', 2K + 2iK'$) in the elliptic form.

5 Conclusion

The Floquet theory impose strong constraint on the solution for Schrödinger equation with periodic potential. The classical Floquet for real singly-period potential is well understood. Based on the classical theory we present a method to compute the spectral solution when the potential can be treated as a small perturbation. This method can be applied to many smooth periodic potentials, including the elliptic potentials along the period $2\omega_1$.

The classical Floquet theory is unable to deal with potentials of elliptic function; the precise relation of multi-periods $2\omega_1, 2\omega_2, 2\omega_3$ and spectral theory is unexplained. We studied this problem for the Lamé potential in [19], and related monodromy along all periods to all possible asymptotic solutions. In this paper we extended our previous work by applying this method to the ellipsoidal wave equation. We reproduced an already known spectral solution in (30) and obtained two new solutions in (20) and (36). The solutions presented in the Section 4 are based on computational results, we do not have a rigorous mathematical theory

for them. Their connection, if there is any, to the previous study [23, 24, 25, 26, 27] remains unclear.

Another interesting point is that, for the elliptic potentials we have examined, the q -dependence of the eigenvalues is expressed by (quasi)modular functions. The modular functions ζ_1, g_2, g_3 and e_i in the formulae (19), (25), and the elliptic module k in the formulae (30), (36), all can be rewritten in terms of the theta functions.

In retrospect, the doubly-periodic Floquet theory is the complete theory, even for a singly-periodic potential it gives a more complete explanation. For example, the Mathieu equation with potential $u(x) = 2\theta_1 \cos 2x$ has three asymptotic spectral solutions [19]. The large λ expansion is explained in the context of singly-periodic Floquet theory as in the Sections 2 and 3, but the other two asymptotic expansions lack such an explanation. Now we know that the other two expansions are special limit of the corresponding asymptotic solutions of the equations with elliptic potential given in the Section 4. In the limit involving $k \rightarrow 0$ the quarter period $iK' \rightarrow i\infty$, therefore we lose the trace of periods $2iK'$ and $2K + 2iK'$.

Appendix A The contour integrals of \mathcal{I}_m and \mathcal{J}_m

When we use the Jacobian form of the elliptic potential to compute the monodromies, we need to perform integrals $\mathcal{I}_m = \int \text{sn}^m z dz$ and $\mathcal{J}_m = \int \text{cn}^m z dz$, for integers m , along three periodic paths of the function $\text{sn}^2 z$. Using the following recursion relations we could reduce the problem to computation of the first few integrals for $m = \pm 1, \pm 2$. The recursion relations for the indefinite integrals are [30]

$$(m+1)\mathcal{I}_m - (m+2)(1+k^2)\mathcal{I}_{m+2} + (m+3)k^2\mathcal{I}_{m+4} - \text{sn}^{m+1}z \text{cn}z \text{dn}z = 0, \quad (38)$$

$$(m+1)k'^2\mathcal{J}_m - (m+2)(1-2k^2)\mathcal{J}_{m+2} - (m+3)k^2\mathcal{J}_{m+4} + \text{cn}^{m+1}z \text{sn}z \text{dn}z = 0. \quad (39)$$

The definite integrals are performed along trajectories bounded inside the period rectangle $[z_0, z_0 + 2K] \times [z_0, z_0 + 2iK']$ in the z -plane. The integral trajectories are not closed, the endpoints of the trajectories differ by the three period vectors $2K, 2iK', 2K + 2iK'$. For $m = \pm 1$ we use the integral formulae [30]

$$\int \text{sn}z dz = \frac{1}{2k} \ln \frac{\text{dn}z - k \text{cn}z}{\text{dn}z + k \text{cn}z}, \quad \int \frac{dz}{\text{sn}z} = \frac{1}{2} \ln \frac{\text{dn}z - \text{cn}z}{\text{dn}z + \text{cn}z}, \quad (40)$$

$$\int \text{cn}z dz = -\frac{1}{2ik} \ln \frac{\text{dn}z - i k \text{sn}z}{\text{dn}z + i k \text{sn}z}, \quad \int \frac{dz}{\text{cn}z} = -\frac{1}{2k'} \ln \frac{\text{dn}z - k' \text{sn}z}{\text{dn}z + k' \text{sn}z}. \quad (41)$$

The logarithm expressions in (40) and (41) have branch cuts, we need to choose the correct integral trajectories in the z -plane so that the corresponding paths of $\text{cd}z = \text{cn}z/\text{dn}z$ and $\text{sd}z = \text{sn}z/\text{dn}z$ cross the branch cuts in a correct manner. For the $2iK'$ periodic integrals

(40) a trajectory is chosen to ensure the path of cdz does not cross the branch cut $[-\frac{1}{k}, \frac{1}{k}]$ but does cross the branch cuts $[-1, +1]$. In this way, we get the correct values $I_{+1} = 0, I_{-1} = i\pi$ used in the Section 4. In a similar way, a trajectory is chosen for the $2K + 2iK'$ periodic integrals (41) to obtain $J_{+1} = 0, J_{-1} = -i\pi\frac{1}{k}$.

The contour integrals can be explained in another way. We change the variable by $\text{sn}^2 z = \xi$, whose inverse $z = \text{sn}^{-1} \xi$ is a complex version of the Schwarz-Christoffel mapping. A quarter of period rectangle in the z -plane is mapped onto half of the ξ -plane. By analytical continuation, the whole period rectangle is mapped twice onto the ξ -plane, therefore a periodic trajectory in the z -plane is mapped to a closed contour in the ξ -plane. We denote the contours in the ξ -plane by α, β, γ , respectively.

The contour α is related to the large energy perturbation, the computation in the Jacobian form is carried as follows. The KdV Hamiltonian densities $v_{2\ell-1}$ only contain $\text{sn}^m z$ for even $m \in 2\mathbb{Z}_+$. Using the recursion relation of \mathcal{I}_m , we only need to perform the integral

$$\int_{z_0}^{z_0+2K} \text{sn}^2 z dz = \frac{1}{2} \oint_{\alpha} \frac{\sqrt{\xi} d\xi}{\sqrt{(1-\xi)(1-k^2\xi)}} = \frac{\pi}{2} {}_2F_1\left(\frac{1}{2}, \frac{3}{2}, 2; k^2\right) = \frac{2(K-E)}{k^2}, \quad (42)$$

where $K = K(k^2)$ and $E = E(k^2)$ are the complete elliptic integrals of the first and the second kind. There are four branch points at $\xi = 0, 1, \frac{1}{k^2}$ and ∞ , the branch cuts are between pairs of branch points, as shown in Figure 1. To compare with the results in the Section 3, we need to use a relation between elliptic functions,

$$\zeta_1 = (e_1 - e_2) \frac{E}{K} - e_1. \quad (43)$$

To prove this relation some identities of elliptic functions are needed, see more details in the Appendix B.

The contours β and γ are related to the small energy expansions discussed in the Section 4, where the definite integrals I_m and J_m , for odd $m \in 2\mathbb{Z} + 1$, are used. Using the recursion relations (38), (39) we only need to perform the integrals of $I_{\pm 1}$ and $J_{\pm 1}$. The contours β and γ are shown in Figure 2 and Figure 3, they are chosen to avoid crossing the branch cuts. We draw both contours β and γ with one side stretched to far away because such periodic trajectories in the z -plane typically would pass through the neighbourhood of poles of $\text{sn} z$.

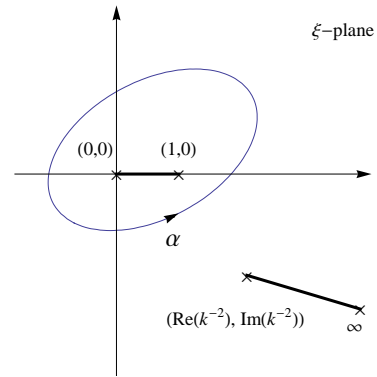


Figure 1: Integral contour α .

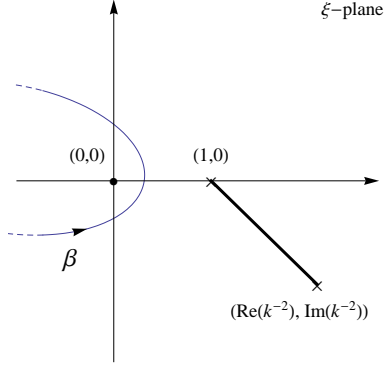


Figure 2: Integral contour β for I_{-1} .

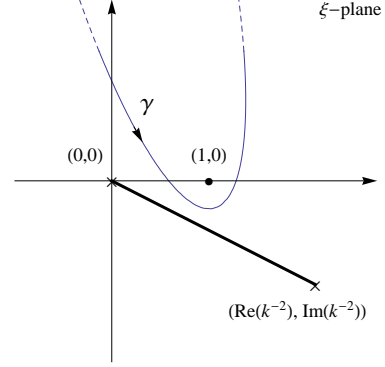


Figure 3: Integral contour γ for J_{-1} .

The integrals $I_{\pm 1}$ are

$$I_1 = \frac{1}{2} \oint_{\beta} \frac{d\xi}{\sqrt{(1-\xi)(1-k^2\xi)}}, \quad I_{-1} = \frac{1}{2} \oint_{\beta} \frac{d\xi}{\xi \sqrt{(1-\xi)(1-k^2\xi)}}. \quad (44)$$

There are two branch points at $\xi = 1$ and $\xi = \frac{1}{k^2}$ for $I_{\pm 1}$, the branch cut is between the branch points. There is a pole at $\xi = 0$ for I_{-1} . Then only I_{-1} receives non-vanishing residue $i\pi$ at $\xi = 0$.

The integrals $J_{\pm 1}$ are

$$J_1 = \frac{1}{2} \oint_{\gamma} \frac{d\xi}{\sqrt{\xi(1-k^2\xi)}}, \quad J_{-1} = \frac{1}{2} \oint_{\gamma} \frac{d\xi}{(1-\xi)\sqrt{\xi(1-k^2\xi)}}. \quad (45)$$

Now there are two branch points at $\xi = 0$ and $\xi = \frac{1}{k^2}$ for $J_{\pm 1}$, the branch cut is between the branch points. There is a pole at $\xi = 1$ for J_{-1} . Therefore only J_{-1} receives non-vanishing residue $-i\pi \frac{1}{k}$ at $\xi = 1$.

Appendix B Some formulae for elliptic functions

In this Appendix we normalize the convention for some elliptic function formulae useful in this paper, and prove the relation (43). The references are [1, 2, 6, 8, 30].

All the elliptic functions in our study can be expressed in terms of the Jacobi elliptic theta functions. The theta functions are represented by the series

$$\begin{aligned} \vartheta_1(\chi, q) &= -iq^{\frac{1}{8}} \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(n+1)}{2}} e^{i(2n+1)\chi}, & \vartheta_2(\chi, q) &= q^{\frac{1}{8}} \sum_{n=-\infty}^{\infty} q^{\frac{n(n+1)}{2}} e^{i(2n+1)\chi}, \\ \vartheta_3(\chi, q) &= \sum_{n=-\infty}^{\infty} q^{\frac{n^2}{2}} e^{2in\chi}, & \vartheta_4(\chi, q) &= \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n^2}{2}} e^{2in\chi}. \end{aligned} \quad (46)$$

The theta constants are $\vartheta_r = \vartheta_r(q) = \vartheta_r(0, q)$, and the χ -derivatives of theta functions are $\vartheta'_r = \vartheta'_r(0) = \partial_\chi \vartheta_r(\chi, q)|_{\chi \rightarrow 0}$ and similar for higher order derivatives, where $r = 1, 2, 3, 4$.

The Weierstrass elliptic function can be expressed by

$$\wp(x; 2\omega_1, 2\omega_2) = \left(\frac{\pi}{2\omega_1}\right)^2 \left(-\partial_\chi^2 \ln \vartheta_1(\chi, q) + \frac{\vartheta_1'''}{3\vartheta_1'} \right), \quad (47)$$

with $\chi = \frac{\pi x}{2\omega_1}$ and $q = \exp 2\pi i \tau = \exp 2\pi i \frac{\omega_2}{\omega_1}$. The constant term on the right-hand side is $\zeta_1 = -\frac{\pi^2}{12\omega_1^2} \frac{\vartheta_1'''}{\vartheta_1'}$. By the following relations of theta functions,

$$\begin{aligned} \vartheta_1(\chi + \frac{1}{2}\pi, q) &= \vartheta_2(\chi, q), & \vartheta_1(\chi + \frac{1}{2}\pi\tau, q) &= iq^{-\frac{1}{8}} e^{-i\chi} \vartheta_4(\chi, q), \\ \vartheta_1(\chi + \frac{1}{2}\pi + \frac{1}{2}\pi\tau, q) &= q^{-\frac{1}{8}} e^{-i\chi} \vartheta_3(\chi, q), \end{aligned} \quad (48)$$

the functions $\wp(x + \omega_i; 2\omega_1, 2\omega_2)$ with $i = 1, 2, 3$ can be expressed in the same form as in (47) only with $\vartheta_1(\chi, q)$ in the logarithm substituted by $\vartheta_2(\chi, q), \vartheta_4(\chi, q), \vartheta_3(\chi, q)$, respectively. There is a set of familiar relations about $e_i(q)$ and $\vartheta_r(q)$,

$$e_1 = \frac{\pi^2}{12\omega_1^2} (-\vartheta_2^4 + 2\vartheta_3^4), \quad e_2 = \frac{\pi^2}{12\omega_1^2} (-\vartheta_2^4 - \vartheta_3^4), \quad e_3 = \frac{\pi^2}{12\omega_1^2} (2\vartheta_2^4 - \vartheta_3^4). \quad (49)$$

The modular invariants g_2, g_3 are expressed by theta constants through $g_2 = -4(e_1 e_2 + e_1 e_3 + e_2 e_3)$, $g_3 = 4e_1 e_2 e_3$, the Eisenstein series are given by $\zeta_1 = \frac{1}{3}(\frac{\pi}{2\omega_1})^2 E_2$, $g_2 = \frac{4}{3}(\frac{\pi}{2\omega_1})^4 E_4$, $g_3 = \frac{8}{27}(\frac{\pi}{2\omega_1})^6 E_6$.

In the formula (47), we have the value e_i on the left-hand side by setting $x = \omega_1, \omega_2, \omega_3$. On the right-hand side the corresponding expressions can be simplified by the fact $\vartheta_1' \neq 0, \vartheta_2' = \vartheta_3' = \vartheta_4' = 0$. We get another set of useful relations expressing e_i in terms of theta functions,

$$e_1 = \left(\frac{\pi}{2\omega_1}\right)^2 \left(-\frac{\vartheta_2''}{\vartheta_2} + \frac{\vartheta_1'''}{3\vartheta_1'}\right), \quad e_2 = \left(\frac{\pi}{2\omega_1}\right)^2 \left(-\frac{\vartheta_4''}{\vartheta_4} + \frac{\vartheta_1'''}{3\vartheta_1'}\right), \quad e_3 = \left(\frac{\pi}{2\omega_1}\right)^2 \left(-\frac{\vartheta_3''}{\vartheta_3} + \frac{\vartheta_1'''}{3\vartheta_1'}\right). \quad (50)$$

Define the differential operator $\mathcal{D} = q\partial_q = \frac{1}{2\pi i}\partial_\tau$. Using the heat equation $i\pi\partial_\chi^2 \vartheta_r(\chi, q) + 4\partial_\tau \vartheta_r(\chi, q) = 0$, we have $-\frac{\vartheta_r''}{\vartheta_r} = 2\mathcal{D} \ln \vartheta_r^4$. Then the relations (50) is equivalent to the following relations,

$$\mathcal{D} \ln \vartheta_2^4 = \frac{2\omega_1^2}{\pi^2} (\zeta_1 + e_1), \quad \mathcal{D} \ln \vartheta_4^4 = \frac{2\omega_1^2}{\pi^2} (\zeta_1 + e_2), \quad \mathcal{D} \ln \vartheta_3^4 = \frac{2\omega_1^2}{\pi^2} (\zeta_1 + e_3). \quad (51)$$

The last set of identities we need are the relations of the complete elliptic integrals K, E and the theta constants,

$$K = \frac{\pi}{2} \vartheta_3^2, \quad \frac{E}{K} = k'^2 \left(1 + \frac{d \ln K}{d \ln k}\right). \quad (52)$$

Taking into account the relation of k and q given in (16), we have

$$\frac{d \ln K}{d \ln k} = \frac{\mathcal{D} \ln \vartheta_3^4}{\mathcal{D} \ln \vartheta_2^4 - \mathcal{D} \ln \vartheta_3^4} = \frac{\zeta_1 + e_3}{e_1 - e_3}. \quad (53)$$

Then the relation (43) is proved.

The Jacobian elliptic functions can be expressed by the theta functions in a similar way,

$$\operatorname{dn}^2(z|k^2) = \partial_z^2 \ln \vartheta_4\left(\frac{\pi z}{2K}, q\right) + \frac{E}{K}, \quad (54)$$

from which the expressions for $\operatorname{sn} z$, $\operatorname{cn} z$ can be derived. To verify the consistency of (47) and (54), we should use the relation (15),(48),(49), and (52),(53). The Jacobi zeta function is also useful for future study, it is defined by

$$\operatorname{zn}(z|k^2) = \partial_z \ln \vartheta_4\left(\frac{\pi z}{2K}, q\right). \quad (55)$$

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